

Uniqueness of the Seiberg-Witten Effective Lagrangian

R. Flume¹, M. Magro², L.O'Raifeartaigh, I. Sachs and O. Schnetz
*Dublin Institute for Advanced Studies,
10 Burlington Road, Dublin 4, Ireland.*

Abstract

The low energy effective Lagrangian for $N = 2$ supersymmetric Yang-Mills theory, proposed by Seiberg and Witten is shown to be the unique solution, assuming only that supersymmetry is unbroken and that the number of strong-coupling singularities is finite. Duality is then a consequence rather than an input.

1 Introduction

Over the last two years considerable progress in the understanding of the strong coupling regime in $N = 2$ Yang-Mills theory has been made, pioneered by the work of Seiberg and Witten [1] where a self-consistent non-perturbative superfield effective Lagrangian was found for an $SU(2)$ gauge group. This has later been extended to higher groups [2]. The crucial properties which make the $N = 2$ -theory accessible for an exact treatment are the holomorphic properties of the effective Lagrangian and the 1-loop exactness of the perturbative β -function. The idea is then to extrapolate the perturbative (weak coupling) β -function to the full range of couplings. Making the assumption that the strong coupling behaviour is related in a definite way (ie. by S -duality) to the weak coupling regime lead to an elegant, self-consistent solution for the low energy effective Lagrangian in [1]. Furthermore the duality is closely related to the electric-magnetic duality conjectured earlier by Olive and Montonen [3] to be a non-perturbative property of some quantum field theories.

The authors of [1] motivate the duality assumption with several convincing arguments based on physical intuition. Furthermore recent explicit 1-and 2-instanton computations [4, 5, 6, 7] confirm the first two coefficients in the asymptotic expansion of the exact solution proposed in [1]. On the other hand strongly coupled systems have surprised us on several occasions with counter-intuitive results. Also, recent instanton calculations indicate that the extension of [1] for models including matter multiplets given in [8] is ambiguous [9]. Therefore the question whether the assumptions made in [1, 8] are necessary or whether the exact effective Lagrangian can be obtained from

¹Volkswagen Foundation Fellow, on leave of absence University of Bonn

²On leave of absence from ENSLAPP, Groupe de Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France

weaker assumptions is of much interest. It is that question we address here. Indeed we prove that, provided supersymmetry is unbroken, the uniqueness of SW solution follows assuming only that the number of strong-coupling singularities on the moduli space is finite.

Our strategy is as follows: we first construct the general effective Lagrangian, compatible with perturbation theory, analyticity and the θ -vacuum. Note that the first two conditions are consequences of supersymmetry alone. It turns out that the existence of the θ -parameter restricts the set of admissible solutions considerably. Indeed the existence of the θ -parameter already contains the seeds of duality, in the sense that either the theory has complete $PSL(2, \mathbf{Z})$ -symmetry, or the moduli space contains a point which is conjugate to the weak coupling regime. Even so, the solution is not unique at that level. We then make use of the fact that for an asymptotically free theory, the scale of the low energy coupling is set by the mass of the lightest charged field. We therefore impose the further constraint that the mass of the lightest charged field be finite except in the asymptotically free regime. The set of solutions then collapses to a single member, which is precisely the Seiberg-Witten solution. The same result is achieved by demanding that the expectation value $\text{Tr}\langle\phi^2\rangle$, where ϕ is the scalar component of the $N=2$ -multiplet, is finite except in the asymptotically free regime. This property is very much expected for an asymptotically free theory. We also obtain the dependence of this observable on the point in the moduli space on general grounds.

The plan of the paper is as follows: In section 2 we review the peculiar properties of $N=2$ Yang-Mills theory which we will take as the only inputs for the later sections. In section 3 we construct the general solution for the effective Lagrangian compatible with these requirements. In section 4 we then include the finite mass constraint in our analysis and show that it reduces the above set of solutions to the SW one. The role of the expectation value $\text{Tr}\langle\phi^2\rangle$ is explained in section 5. Some of the mathematical constructions needed for the main text and the argument relating the θ -vacuum to duality are given in two appendices.

2 Review of $N=2$ Yang-Mills

A crucial property of $N=2$ -Yang-Mills theory is the presence of flat directions in the potential for the scalar component ϕ of the $N=2$ -multiplet, $V(\phi) = \frac{1}{g^2} \text{Tr}[\phi, \phi^\dagger]^2$, where g is the coupling constant. The potential vanishes for constant ϕ taking its value in the Cartan subalgebra of the gauge group. Furthermore, for unbroken supersymmetry, this degeneracy cannot be removed by quantum corrections [10]. For $\phi \neq 0$, the Higgs mechanism breaks the gauge symmetry spontaneously down to $U(1)^l$, where l is the rank of the Cartan subalgebra. In what follows we concentrate on $SU(2)$. As explained in [11, 12], it can be deduced from $N=2$ supersymmetry that, when expressed in terms of the Cartan algebra-valued $N=2$ -superfield $\mathcal{A} = \phi + \theta\chi + \dots$, the most general local $N=2$ supersymmetric low energy effective Lagrangian must be of the form

$$\Gamma[\mathcal{A}] = \frac{1}{4\pi} \text{Im} \int d^4x d^2\theta_1 d^2\theta_2 \mathcal{F}(\mathcal{A}), \quad (1)$$

where the prepotential \mathcal{F} , to be determined, is the result of integrating out the massive (ie. charged, root-valued) fields. In $N=1$ notation (1) becomes [13]

$$\Gamma[A, W_\alpha] = \frac{1}{8\pi} \text{Im Tr} \int d^4x \left\{ \int d^2\theta d^2\bar{\theta} (A_D \bar{A} - \bar{A}_D A) + \int d^2\theta \tau(A) W^\alpha W_\alpha \right\}, \quad (2)$$

where A and W_α are the chiral- and vector $N=1$ superfields respectively. Furthermore

$$A_D = \mathcal{F}'(A) \quad \text{and} \quad \tau(A) = \mathcal{F}''(A) = A'_D(A). \quad (3)$$

Since $\tau(A)$ is the coefficient of the kinetic term in (2) its imaginary part must be positive. On the other hand its real part plays the role of an effective θ -angle: $\text{Re } \tau = \frac{\theta}{2\pi}$. Thus a shift of θ by 2π corresponds to $\tau \mapsto T(\tau) = \tau + 1$. Therefore the group $\mathcal{T} = \{T^n, n \in \mathbb{Z}\}$ is a symmetry group of the theory. The invariance of the chiral part in (2) together with (3) requires then that \mathcal{T} be represented linearly on (A, A_D) by a subgroup of $U(1) \times \mathcal{T}$.

A remarkable consequence of $N=2$ supersymmetry is that the mass of the charged particles must be proportional to the central charge Z of the SUSY-algebra. More precisely [1, 14]

$$M = \sqrt{2}|Z| \quad \text{where} \quad Z = \mathbf{n} \cdot \mathbf{a}, \quad \mathbf{a} = \begin{pmatrix} a_D \\ a \end{pmatrix} \quad \text{and} \quad \mathbf{n} = (n_m, n_e) \neq 0. \quad (4)$$

Here $a = \text{Tr}(\langle \phi \rangle \sigma_3)$ is the expectation value of the scalar component of the $N=2$ superfield and $a_D = \mathcal{F}'(a)$. The integers n_e and n_m label electric and magnetic charge respectively. The use of the dual variable a_D exhibits the $SL(2, \mathbf{Z})$ invariance of the mass formula (4) and of the first integral in (2).

At the classical level $a_D = a\tau$. Furthermore the $U(1)$ -invariance of the first integral in (2) reflects the R -symmetry of the Lagrangian. The theory is then parameterized by two real parameters, g^2 and $|a|$ (the phase of a is irrelevant because of R -symmetry). In the quantum theory the mass of the charged fields sets the scale for the low energy coupling. Because of asymptotic freedom perturbation theory is then valid for large masses ($M \gg \Lambda$). At the semiclassical level we then have ($\Lambda=1$) [15]

$$\tau(a) = \frac{i}{\pi} (\log(a^2) + c) \quad (5)$$

where c depends on the renormalization scheme adopted. The divergence of the R -current is in the same supermultiplet as the trace of the energy-momentum tensor and is therefore also anomalous at 1-loop. Consequently R -symmetry is broken to the discrete group $a \rightarrow e^{\frac{i\pi}{2}} a$. Because of the $N=2$ supersymmetry higher loop perturbative corrections to the running coupling are absent [16]. Note that due to the quantum corrections the low energy theory is parameterized by either of the complex parameters a with values in \mathbf{C} , or τ which takes any value in the upper half plane H . In particular, the space of inequivalent vacua, or moduli space, \mathcal{M} is one (complex) dimensional.

3 Determination of \mathcal{F}

3.1 Statement of the Problem

In addition to the perturbative corrections, reviewed in the last section, the low energy effective theory receives corrections due to topologically non-trivial configurations [17, 18, 11]. In principle, of course, $\mathcal{F}(a)$ could be computed directly from the functional integral but in practice this is far too difficult. The problem is then to determine $\mathcal{F}(a)$ or equivalently $\tau(a)$ from their properties established in the previous section: τ is an analytic function of a with $\text{Im}(\tau(a)) \geq 0$ and satisfies the boundary condition (5). The analyticity of τ follows from the analyticity of the prepotential \mathcal{F} .

The $SL(2, \mathbf{Z})$ structure mentioned above and the structure of the instanton contributions to (5) found in [1, 11] suggest that, asymptotically at least, $\tau(a)$ is an inverse modular function. Seiberg and Witten proposed that τ be an inverse modular function everywhere on \mathcal{M} , where \mathcal{M} is parameterized by some variable $u \in \mathbf{C}$, with $u \rightarrow a^2$ asymptotically. To go further they made the two following assumptions:

- (i) *Minimality*: $\tau(u)$ has just *two* singularities for finite u ,
- (ii) *Duality*: the monodromy matrix for $\tau(u)$ at one of the singularities is the transpose of that for $u \rightarrow \infty$.

They then showed that there is a unique function $\tau(u)$ with these properties, which in turn can be lifted to a unique effective prepotential $\mathcal{F}(a)$. Later it was found [4, 5, 6, 7] that the first two coefficients in the asymptotic expansion of the proposed $\mathcal{F}(a)$ agreed with direct instanton computations. The latter result is evidently a strong indication that the SW Ansatz is correct. More recently it has been shown that $\mathcal{F}(a)$ can be obtained from somewhat weaker assumptions. However, a critical assumption that is made in [19] is that $\text{Tr}(\langle \phi^2 \rangle)$ parameterizes the moduli space \mathcal{M} . As we shall see this is equivalent to assuming that the Wronskian of a and a_D with respect to the moduli parameter is constant.

Below we show that these assumptions, although correct, are not necessary. Specifically our inputs are:

- (a) The effective coupling constant $\tau = \frac{\theta_{eff}}{2\pi} + i4\pi g_{eff}^{-2}$ takes all values in the upper half plane H .
- (b) The mass m of the lightest charged field (possibly composite) is finite except in the asymptotically free region.
- (c) The mass M is a single valued function on the moduli space, $M = M(P)$, $P \in \mathcal{M}$.
- (d) The set of singular points of \mathcal{M} is finite.

3.2 Uniformization

Due to the 1-loop corrections (5), Z , a_D and τ are multiple-valued transcendental functions of a , whereas Z , a_D and a are single-valued functions of the coupling $\tau \in H$. Thus, τ is the obvious candidate for the uniformizing parameter. Of course, distinct values of the effective coupling τ may correspond to equivalent vacua. For instance, we already know that τ and $\tau+1$ have to be identified. We therefore introduce the concept

of a *maximal equivalence group* G . This is defined as the group of all transformations g of τ that leave $P(\tau) \in \mathcal{M}$ invariant³ i.e.

$$G = \{g / P(g\tau) = P(\tau), \forall \tau \in H\}. \quad (6)$$

The action of G on τ then induces an action \tilde{G} on $\mathbf{a}(\tau)$. The invariance of the spectrum (4) together with the integer valuedness of the charges (n_m, n_e) imply that \tilde{G} acts linearly on \mathbf{a} and indeed is a subgroup of $U(1) \times PSL(2, \mathbf{Z})$. The conditions $\text{Im}\tau \geq 0$ and

$$\tau(a) = \frac{da_D}{da}, \quad (7)$$

then imply that G is a subgroup of $PSL(2, \mathbf{Z}) = SL(2, \mathbf{Z})/\mathbf{Z}_2$ and acts on τ by modular transformations. As noted in the context of the SW solution in [21], the $U(1)$ factor in \tilde{G} cannot be ignored because the mass-spectrum is determined by $|Z|$ and not Z . As we shall see in the next section, and was noted for the SW solution in [22], the $U(1)$ -factor corresponds to the fact that \mathbf{a} is a section of a non-trivial $U(1) \times G$ bundle. We conclude that the set of inequivalent couplings τ is a fundamental domain $D = H/G$ of the maximal equivalence group G . The domain D is then in 1–1 correspondence with the moduli space \mathcal{M} . Because $G \subset PSL(2, \mathbf{Z})$, D is a polygon bounded by arcs [23].

It is at this point that we make use of the assumption that the number of strong coupling singularities be finite. The corners of the fundamental domain D correspond to singularities in the moduli space. Finite number of singularities then requires that the polygon D has a finite number of corners. If this condition was not fulfilled then the set of corners of D and therefore the set of singularities in \mathcal{M} would necessarily have an accumulation point.

3.3 General Solution

In this section we show that for any group G for which the fundamental domain has a finite number of corners, we can construct functions $a(\tau)$ and $a_D(\tau)$ such that $\text{Im}\tau \geq 0$,

$$\tau = \frac{da_D}{da} \quad \text{and} \quad |\mathbf{n} \cdot \mathbf{a}(g\tau)| = |\mathbf{n}' \cdot \mathbf{a}(\tau)| \quad \text{for} \quad g \in G. \quad (8)$$

General Form of τ : The polygon D is bounded by arcs and has a finite number of corners. One of them corresponds to the weak-coupling singularity $\tau = i\infty$. We now make use of the fact that D can be parameterized by means of a Fuchsian mapping [24] $\tau : z \in H \mapsto \tau(z) \in D$. The Schwarzian of τ

$$\{\tau, z\} \equiv \frac{\tau'''}{\tau'} - \frac{3}{2} \left(\frac{\tau''}{\tau'} \right)^2 \quad (9)$$

³Strictly speaking, this group should be defined on the level of \mathbf{a} rather than τ ; however, this distinction only becomes important when matter hypermultiplets are included [20].

has the simple form

$$\{\tau, z\} = \sum_{i=1}^n \left[\frac{1}{2} \frac{1 - \alpha_i^2}{(z - a_i)^2} + \frac{\beta_i}{z - a_i} \right], \quad (10)$$

where $n + 1$ is the number of edges of D , the a_i 's are the points on the real axis into which the corners of the polygon D are mapped. Furthermore the $\alpha_i \in [0, 1)$ are the interior angles of the polygon. In (10) we have chosen to map the weak-coupling singularity $\tau = i\infty$ to infinity in the z -plane. Since the corresponding angle is zero we have [24]

$$\{\tau, z\} \rightarrow \frac{1}{2z^2} \quad \text{for } z \rightarrow \infty. \quad (11)$$

This condition puts two constraints on the β 's. Thus there are $3n - 2$ parameters for each set of polygons with n corners. Actually only $3n - 4$ of these are independent because the origin and the scale of z on the real axis are still free parameters. If we furthermore identify segments on the real line in the z -plane, that are mapped into edges of the polygon D which are equivalent with respect to G , then the equivalence group G of τ is just its monodromy group with respect to z . This makes it natural to parameterize the moduli space by $z \in H$.

The useful property of Fuchsian functions is that $\tau(z)$ can be written as [24]

$$\tau(z) = \frac{y_1(z)}{y_2(z)}, \quad (12)$$

where $\mathbf{y} = (y_1, y_2)$ is solution of the second order differential equation ($' = \frac{d}{dz}$)

$$\mathbf{y}'' + Q\mathbf{y} = 0 \quad \text{with} \quad Q(z) = \frac{1}{2}\{\tau, z\}. \quad (13)$$

Since the Wronskian $W(y_1, y_2) = y_1'y_2 - y_2'y_1$ of y_1 and y_2 is constant G acts linearly on \mathbf{y} with a trivial $U(1)$.

Lifting: We now show that the previous second order differential equation for \mathbf{y} can be lifted to a second order differential equation for \mathbf{a} . This may come as a surprise as the lifting from τ to (a_D, a) involves an integration leading *a priori* to a third order differential equation for \mathbf{a} . However, we show below that it is always possible to lift (13) to a second order differential equation for \mathbf{a} provided we allow for an extra $U(1)$ -factor to appear in the action of G on \mathbf{a} .

Define \mathbf{a} by

$$\mathbf{a} = f'\mathbf{y} - f\mathbf{y}' = \mathbf{W}(f, \mathbf{y}), \quad (14)$$

where f is any section of the $U(1)$ -bundle and $\mathbf{W}(f, \mathbf{y})$ is the Wronskian of f and \mathbf{y} . Differentiating (14) with respect to z gives

$$\mathbf{a}' = f''\mathbf{y} - f\mathbf{y}'' = (f'' + Qf)\mathbf{y} \quad (15)$$

and thus \mathbf{a} satisfies (8). Furthermore it is easy to see from (13) and (14) that \mathbf{a} satisfies a second order differential equation.

The construction (14) is in fact the only way to satisfy (8). Indeed, from (8) we conclude that

$$\mathbf{a}' = g\mathbf{y}, \quad (16)$$

where g is some $U(1)$ -section. The integral of (16) is precisely given by (14) with f satisfying the inhomogeneous form of (13)

$$f'' + Qf = g. \quad (17)$$

The constant of integration must be set to zero for \mathbf{a} to transform homogeneously with respect to G . Note that the action of G on \mathbf{a} is also that of monodromy transformations in z .

Boundary Conditions: To complete the construction we need to satisfy the boundary conditions given by the semiclassical contribution (5). We recall from (11) and (13) that

$$Q(z) \simeq \frac{1}{4z^2} \quad \text{for } z \rightarrow \infty. \quad (18)$$

Therefore

$$\mathbf{y} \propto z^{\frac{1}{2}}(1, \ln(z)) \quad \text{for } z \rightarrow \infty \quad (19)$$

and hence

$$\tau(z) \rightarrow \frac{ic}{\pi} \ln(z). \quad (20)$$

The constant c is fixed by using the fact that the action of G on τ corresponds to monodromy transformations in z and by requiring that $T : \tau \mapsto \tau + 1$ belongs to G as shown in section 2. First due to the identifications on the real line, for large z , $z \mapsto e^{i\pi}z$ is a monodromy transformation. Under such a transformation $\tau \mapsto \tau + c$. Therefore c is an integer. The condition $T \in G$ leads then to $c = 1$. Compatibility with the semiclassical relation (5) then requires $a \propto z^{\frac{1}{2}}$ for large z , which using (14) and (19) leads to

$$f(z) \propto z \quad \text{for } z \rightarrow \infty. \quad (21)$$

This completes the general construction of the vector $\mathbf{a}(z)$. To summarize we have shown that the action of any equivalence group G for τ can be lifted to the pair of functions $a_D(\tau)$ and $a(\tau)$ and furthermore the action is simply by monodromy transformations on $z \in H$ with proper identifications. Without further constraints the solution is in general not unique. Indeed any Fuchsian function $\tau(z)$ mapping H into a funda-

mental domain of G and any $U(1)$ -section $f(z)$ satisfying the boundary condition (21) is a solution.

4 Finite-Mass Constraint

As discussed in section 2, infinite mass for all (possibly composite) charged fields implies that the full $SU(2)$ -theory is weakly coupled due to asymptotic freedom. Therefore consistency requires that the mass m of the lightest charged field diverges only in the perturbative regime. It turns out that the finite mass constraint puts a very strong condition on the set of solutions constructed in the last subsection. It is here also that the extra $U(1)$ -bundle introduced by the function f in (17) becomes important.

4.1 Finite-Mass Condition for f

We first observe that in order to be a $U(1)$ -section $f(z)$ must behave in the vicinity of a singularity z_0 of f as

$$f(z - z_0) \propto (z - z_0)^{r_{z_0}}, \quad (22)$$

(at infinity $f(z) = 1/z^{r_\infty}$).

We now derive a lower bound for each r_z from the finite mass condition. First we recall that the mass spectrum is given by $M = \sqrt{2}|\mathbf{n} \cdot \mathbf{a}|$ and thus using (14) $M = \sqrt{2}|\mathbf{n} \cdot \mathbf{W}(f, \mathbf{y})|$. Finiteness of the mass spectrum requires then that at least one component of $\mathbf{W}(f, \mathbf{y})$ be finite at a given finite point z_0 .

Now, if Q is regular at this point then y_1 and y_2 are also regular and then finiteness of M implies either f is regular at z_0 or f has a singularity at z_0 with $r_{z_0} \geq 1$. If z_0 is a singularity of Q , ie. $z_0 = a_i$, then (13) can be solved locally to give for $\alpha_i \neq 0$

$$y_\pm(z) \propto (z - a_i)^{\frac{1}{2}(1 \pm \alpha_i)}, \quad (23)$$

where y_\pm are the eigenvectors of the monodromy matrix M_{a_i} . Since

$$W(f, y_\pm)(z) \simeq c \left(r_{a_i} - \frac{1}{2}(1 \pm \alpha_i) \right) (z - a_i)^{r_{a_i} - \frac{1}{2}(1 \pm \alpha_i)}, \quad c = \text{const} \quad (24)$$

finiteness of the mass implies

$$r_{a_i} \geq \frac{1}{2}(1 + \alpha_i) \quad \text{or} \quad r_{a_i} = \frac{1}{2}(1 - \alpha_i). \quad (25)$$

For $\alpha_i = 0$, y_1 and y_2 are linear combinations of $z^{\frac{1}{2}}$ and $z^{\frac{1}{2}} \log z$ and the same analysis gives $r_{a_i} \geq \frac{1}{2}$. Thus the condition (25) is also valid for this case. Combining the above results we see that *necessary* conditions for finite M are

$$\begin{aligned} r_z &\geq \frac{1}{2}(1 - \alpha_i) & \text{for} & \quad z = a_i \\ r_z &\geq 1 & \text{for} & \quad z = b_i, \end{aligned} \quad (26)$$

where $\{a_i\}$ are the common singularities of f and Q and $b_i \notin \{a_i\}$ are the points where

f is singular but Q is regular.

4.2 Total Residue Condition

To proceed further we need a general theorem on residues, namely that the total residue i.e. the sum of the residues [25] of a meromorphic form ω on a compact manifold is zero,

$$\sum_i \oint_{C_i} \omega = 0, \quad (27)$$

where the integrations are taken along closed curves C_i associated with any triangulation of the manifold. Applying this result to the moduli space \mathcal{M} , the form $\frac{df}{f}$, which is meromorphic because f is a section of a line bundle, and a triangulation of \mathcal{M} around all inequivalent poles and zeros of f , we get

$$\sum_{z \text{ interior}} r_z + \frac{1}{2} \sum_{z \in \mathbb{R}} r_z + \frac{1}{2} r_\infty = 0. \quad (28)$$

The $\frac{1}{2}$'s in (28) are due to the fact that singular points on the real line are pairwise identified.

4.3 Resolution of the Finite Mass Condition

An immediate consequence of (28) is that f cannot have a singularity at a point where Q is regular because a single singularity of this kind would already saturate (28) with $r_\infty = -1$ from (21) and $\tau(z)$ has at least two singularities. Hence we can restrict ourself to the first possibility in (26). Using (28) we then obtain

$$n - \sum_{i=1}^n \alpha_i \leq 2. \quad (29)$$

The l.h.s. of (29) is directly related to the index μ of G in $PSL(2, \mathbf{Z})$. The index μ is the order of the coset $PSL(2, \mathbf{Z})/G$. To see the connection we endow the upper half plane H with the $PSL(2, \mathbf{Z})$ -invariant metric [26] $(\text{Im } \tau)^{-2} d\tau d\bar{\tau}$. With this metric every copy of the fundamental domain D_0 of $PSL(2, \mathbf{Z})$ has the same area $\pi/3$. Since the fundamental domain D is composed of μ copies of D_0 [23] it has the area $\mu\pi/3$. On the other hand the area of D , which is a polygon bounded by arcs with centers on the real line and which has one zero angle at infinity and n further angles α_i is [26] $\pi(n - 1 - \sum_1^n \alpha_i)$. The equality of the two expressions obtained for the area of D leads to the relation

$$n - \sum_{i=1}^n \alpha_i = \frac{\mu}{3} + 1. \quad (30)$$

Thus the condition (29) is equivalent to $\mu \leq 3$. It is shown in Appendix A, that the only subgroups of $PSL(2, \mathbf{Z})$ with index not greater than 3 and containing T are $\Gamma_0(2)$ and $PSL(2, \mathbf{Z})$ itself. $PSL(2, \mathbf{Z})$ is ruled out for the following reason. Since $\alpha_1 = \alpha_2 = 1/3$ the only f consistent with (28) has $r_{a_1} = 1/3$, $r_{a_2} = 2/3$, or vice versa. However, since a_1 and a_2 correspond to couplings τ that are identified by T , f cannot have different indices at a_1 and a_2 .

For $\Gamma_0(2)$ there are two sets of α_i and r_z :

$$\{\alpha_1 = \alpha_2 = 0\} \quad f(z) = c(z^2 - 1)^{\frac{1}{2}}, \quad (31)$$

$$\{\alpha_1 = \alpha_2 = \frac{1}{2}, \alpha_3 = 0\} \quad f(z) = c(z^2 - 1)^{\frac{1}{4}}(z - a_3)^{\frac{1}{2}}, \quad (32)$$

where we have chosen two singularities a_1 and a_2 to be 1 and -1 . These two solutions correspond to two different choices of the domain D and are related by a coordinate transformation on the moduli space. It can be checked that the physical quantities such as the mass spectrum are scalars under this transformation.

The $\Gamma_0(2)$ solution just discussed is the SW solution. This may be surprising since the group found in [1] is $\Gamma(2)$, whose index in $PSL(2, \mathbf{Z})$ is 6. This difference comes only from the fact that in SW description T plays a distinct role and is not associated to a monodromy transformation of their moduli parameter u . In other words $\Gamma(2)$ is not the *maximal* equivalence group but a subgroup of it. More precisely, $\Gamma(2) = \Gamma_0(2)/\mathbf{Z}_2$. The link between the $\Gamma_0(2)$ and SW descriptions will be given more explicitly at the end of the next section.

5 Physical Description of \mathcal{M}

In this section we wish to consider the expectation value

$$\text{Tr}(\langle \phi^2 \rangle) \quad (33)$$

which, like M has a direct physical meaning. In [7, 27] it was shown that within the instanton approximation

$$\text{Tr}(\langle \phi^2 \rangle) = \pi i (\mathcal{F}(a) - \frac{1}{2} a \mathcal{F}'(a)). \quad (34)$$

Later it was shown in [28] that (34) was generally true as a direct consequence of the superconformal Ward identities. In [1] Seiberg and Witten conjectured that their moduli variable u was identical with $\text{Tr}(\langle \phi^2 \rangle)$. That this is correct can be seen using the result [22] which precisely states that the right hand side of (34) equals u . For this reason and for brevity we define $u \equiv \text{Tr}(\langle \phi^2 \rangle)$. We now discuss the relation between u and the moduli parameter z in the general setting of the previous sections. For this we note from (34) that

$$1 = \frac{i\pi}{2} (a_D \frac{\partial a}{\partial u} - a \frac{\partial a_D}{\partial u}) \quad \Rightarrow \quad \frac{du}{dz} = \frac{i\pi}{2} (a_D a' - a a'_D). \quad (35)$$

Using (16) we then have

$$\frac{du}{dz} = \frac{i\pi}{2} f^2 \left(\frac{f''}{f} + Q \right) (y_1 y_2' - y_2 y_1') = f^2 \left(\frac{f''}{f} + Q \right), \quad (36)$$

where we have used (14) and normalized the Wronskian of y_2 and y_1 to be $\frac{2}{i\pi}$. In order to obtain $u(z)$ we need to integrate this identity ie.

$$u(z) - u(z_1) = \int_{z_1}^z f^2(z') \left(\frac{f''}{f} + Q \right) (z') dz'. \quad (37)$$

This is the required relation between u and z .

Next we show that the Seiberg-Witten solution is the unique solution for which u is finite except for the asymptotic regime, even if we do not insist on the finite mass constraint. For this we note that, using (10), the expression (37) is locally of the form

$$\left\{ \begin{array}{ll} \int_{z_1}^z (z' - a_i)^{2r_{a_i}-2} \left[\frac{1}{4}(1 - \alpha_i^2) + r_{a_i}(r_{a_i} - 1) \right] dz' & \text{for } z \rightarrow a_i \\ \int_{z_1}^z r_{z_0}(r_{z_0} - 1)(z' - z_0)^{2r_{z_0}-2} dz' & \text{for } z \rightarrow z_0 \notin \{a_i\} \end{array} \right\} \quad (38)$$

where we have used $f(z) \propto (z - z_0)^{r_{z_0}}$.

Finiteness of u then leads to the following conditions:

- (i) To every singularity of Q corresponds a zero of f . Furthermore, either $r_{a_i} > \frac{1}{2}$ or $r_{a_i} = \frac{1}{2}(1 - \alpha_i)$.
 - (ii) If f has a singularity where Q is regular ie. $z_0 \notin \{a_i\}$, then $r_{z_0} > \frac{1}{2}$.
- These conditions are rather similar to those obtained from the finite mass constraint. which is, perhaps, not too surprising because of the identity

$$\frac{dZ}{du} = \frac{\mathbf{n} \cdot \mathbf{y}}{f} \quad (39)$$

which follows immediately from (37) and (16) (17). However, they differ in two small respects. First (i) permits a small range $\frac{1}{2}(1 + \alpha_i) > r_{a_i} > \frac{1}{2}$ that is not permitted by the mass condition. Second (ii) requires $r_{z_0} > \frac{1}{2}$ rather than $r_{z_0} \geq 1$. However, when we apply the boundary condition (21) we find that the only solution permitted by (i) and (ii) which is not permitted by the finite mass constraint is

$$r_{a_1} = r_{a_2} = \frac{1}{6}, r_{z_0} = \frac{2}{3} \quad \text{and} \quad \alpha_1 = \alpha_2 = \frac{2}{3}. \quad (40)$$

It is easy to see, however, that the map (40) cannot correspond to any subgroup of $PSL(2, \mathbf{Z})$. Thus $u < \infty$ leads to exactly the same result as the finite mass constraint and therefore leads to a unique solution which is precisely the SW-solution. It then follows that $u = \text{Tr}(\langle \phi^2 \rangle)$ is a good parameter for the moduli space. However, in this paper, in contrast to [1, 22], this property of u is not an assumption but a consequence of the finiteness of m or u itself.

Finally, for $\Gamma_0(2)$ the relation (36) can be explicitly integrated for the two cases (31) and (32). It leads respectively to $u = z$ and $u = \sqrt{1 - z^2}$. Moreover it follows then from (13) and (14) that for both cases $\mathbf{a}(u)$ satisfies the second order differential equation

$$\frac{d^2 \mathbf{a}}{du^2} + \frac{1}{4(u^2 - 1)} \mathbf{a} = 0 \quad (41)$$

which is just the differential equation obtained in [29] for SW solution.

6 Conclusions

We have proved that Seiberg-Witten Ansatz is the unique solution for the low energy effective Lagrangian of $N=2$ -Yang-Mills theory, assuming only that supersymmetry is unbroken and that the number of singularities is finite. In particular the electromagnetic duality is derived, in contrast to the original paper [1] where it was assumed.

En route, we have obtained a construction which lifts any $PSL(2, \mathbf{Z})$ -structure. This construction is straightforward and based on simple differential equations. In particular it does not involve elliptic curves. On the other hand it generalizes the observations in [29, 22] and more recently [30] where the connection with differential equations was explained for the particular case of the Seiberg-Witten solution.

There is no reason why the present construction should not apply to theories including matter hypermultiplets. It therefore has the potential to resolve the puzzles arising there [9]. The question of whether it can be extended to higher groups, where the complex dimension of the moduli space is bigger than 1 is, however, still open.

Acknowledgments

I.S. was partially supported by the Swiss National Science Foundation. M.M. is grateful to the Dublin Institute for Advanced Studies for their kind hospitality. O.S. was supported in parts by the DFG Graduiertenkolleg ‘Starke Wechselwirkung’ and the BMBF. R.F. acknowledges the support of Volkswagenstiftung. I.S. is indebted to F. Hirzenbruch for helpful comments on modular groups.

Appendix A. Modular Groups of Index $\mu \leq 3$

In this Appendix we prove some results on modular groups needed in the main body of the paper. First we recall some basic definitions and results [23].

A modular transformation, is a transformation

$$z \rightarrow \frac{az + b}{cz + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}). \quad (\text{A.1})$$

It is called *elliptic*, *hyperbolic* or *parabolic* corresponding to whether it has two fixed points in the upper half plane, two fixed points on the real axis or one fixed point at ∞ or on the real line. These three cases can equivalently be distinguished by the trace (smaller, bigger or equal to 2).

For our purpose we restrict ourself to the inhomogeneous modular group

$$PSL(2, \mathbf{Z}) = SL(2, \mathbf{Z}) / \pm I \quad \text{generated by} \quad \{T, S\} \quad \text{with} \quad \begin{cases} T : \tau \mapsto \tau + 1 \\ S : \tau \mapsto -\frac{1}{\tau} \end{cases}. \quad (\text{A.2})$$

$H/PSL(2, \mathbf{Z})$ is isomorphic to the Riemann sphere. More generally, if G is a subgroup of $PSL(2, \mathbf{Z})$ of finite index, its fundamental domain $\overline{H/G}$ is isomorphic to a compact Riemann surface [23].

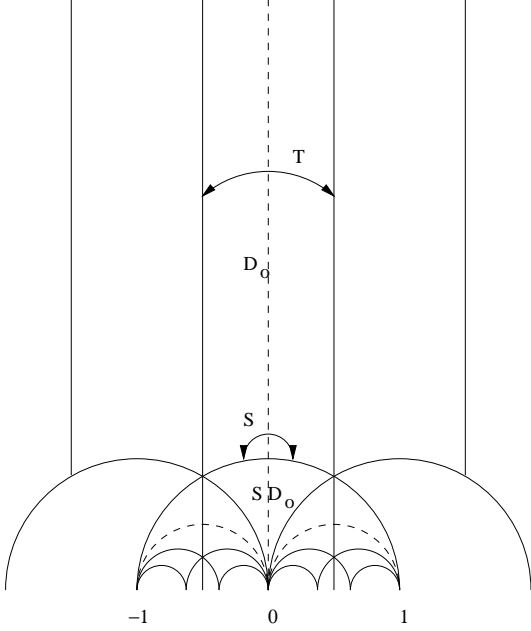
Next we prove the following theorem used in sections 4 and 5.

Theorem *The only subgroups of $PSL(2, \mathbf{Z})$ with index less or equal 3 and containing T are:*

$$\begin{array}{ll} PSL(2, \mathbf{Z}) & \text{generated by } \{T, S\} \\ \Gamma_0(2) & \text{generated by } \{T, ST^2S\} \quad \text{with } ST^2S : \tau \mapsto 1/(-2\tau + 1) \end{array}$$

Proof: First we recall [23] that $S^2 = (ST)^3 = 1$. If $S \in G$ then $G = PSL(2, \mathbf{Z})$ otherwise GS defines one coset of G . In this case $ST \notin G$. Moreover $ST \notin GS$, otherwise $G \ni STS = T^{-1}ST^{-1}$, thus $S \in G$. So GST defines another coset of G and G is of index 3. Consider STS . It has to belong to one of the cosets G , GS or GST . But $STS \notin G$, otherwise $TSTST = S \in G$. Moreover $STS \notin GS$. Hence $STS \in GST \Rightarrow G \ni STST^{-1}S = ST^2ST$. Therefore $ST^2S \in G$ and $G = \Gamma_0(2)$. \square

Figure 1: Fundamental domains for subgroups of the modular group



Appendix B. θ -Vacuum and Duality

In order to discuss the relation between the θ -vacuum and duality, we need the following result on fundamental domains:

Lemma *Let $G \subset PSL(2, \mathbf{Z})$, $G \neq PSL(2, \mathbf{Z})$ such that $T \in G$. Then every fundamental domain D of G has at least one vertex on the real line.*

Proof: Let D_0 be the usual fundamental domain of $PSL(2, \mathbf{Z})$ given in Figure 1. Because $G \neq PSL(2, \mathbf{Z})$, it follows that $S \notin G$. Therefore D contains SD_0 or a copy of SD_0 obtained by applying some element g of G to SD_0 . The image of $0 \in SD_0$

under a modular transformation is either on the real line or at $i\infty$. However, the latter possibility leads to $gSD_0 = T^n D_0$ for some integer n which implies again that $G = PSL(2, \mathbf{Z})$. Hence we conclude that D has a vertex on the real line. \square

Vertices on the real line always correspond to zero angles ie. parabolic substitutions ie. trace 2. From the above it then follows in particular that any solution for the low energy effective Lagrangian which respects the existence of the θ -parameter, either has the monodromy group $PSL(2, \mathbf{Z})$ or otherwise the fundamental domain has at least 1 parabolic vertex in addition to the parabolic point at infinity. The corresponding monodromy, does not commute with the monodromy at infinity. On the other hand it is conjugate (in $PSL(2, \mathbf{Z})$) to the point at infinity. Consequently, there is at least 1 point in the vacuum manifold which is dual (conjugate) to the weak coupling singularity.

References

- [1] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19, (E) **B430** (1994) 485.
- [2] A. Klemm et al., Phys. Lett. **B344** (1995) 169, Int. Jour. Mod. Phys. **A11** (1996) 1929; P.C. Argyres and A.E. Faraggi, Phys. Rev. Lett. **74** (1995) 3931; P.C. Argyres and M.R. Douglas, Nucl. Phys. **B448** (1995) 93; U.H. Danielsson and B. Sundborg, Phys. Lett. **B358** (1995) 273.
- [3] C. Montonen and D. Olive, Phys. Lett. **B72** (1977) 117.
- [4] D. Finnell and P. Pouliot, Nucl. Phys. **B453** (1995) 225.
- [5] N. Dorey, V.V. Khoze and M.P. Mattis, Phys. Rev. **D54** (1996) 2921; hep-th/9607202.
- [6] K. Ito and N. Sasakura, Phys. Lett. **B382** (1996) 95; hep-th/9608054; A. Yung, hep-th/9605096.
- [7] F. Fucito and G. Travaglini, hep-th/9605215
- [8] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484.
- [9] T. Harano and M. Sato, hep-th/9608060; N. Dorey, V. Khoze and M.P. Mattis, hep-th/9611016.
- [10] P. West, Nucl. Phys. **B106** (1976) 219.
- [11] N. Seiberg, Phys. Lett. **B206** (1988) 75.
- [12] B. deWit, M.T. Grisaru and M. Rocek, Phys. Lett. **B374** (1996) 297; A. Pickering and P. West, Phys. Lett. **B383** (1996) 54.
- [13] S.J. Gates, Nucl. Phys. **B238** (1984) 349; G. Sierra and P.K. Townsend, *Supersymmetry and Supergravity* (1983), ed. Milewski, p. 396.
- [14] D. Olive and E. Witten, Phys. Lett. **B78** (1978) 97.

- [15] M.A. Shifman and A.I. Vainshtein, Nucl. Phys. **B277** (1986) 456; A. Gorskii et al., Phys. Lett. **B355** (1995) 466.
- [16] M.T. Grisaru, and W. Siegel, Nucl. Phys. **B201** (1982) 292.
- [17] I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. **B241** (1983) 493.
- [18] V.A. Novikov et al., Nucl. Phys. **B260** (1985) 157.
- [19] G. Bonelli, M. Matone and M. Tonin, hep-th/9610026.
- [20] M. Magro, L.O’Raifeartaigh and I. Sachs, in preparation.
- [21] F. Ferrari and A. Bilal, Nucl. Phys. **B469** (1996) 387, hep-th/9605101.
- [22] M. Matone, Phys. Lett. **B357** (1995) 342.
- [23] R.C. Gunning, *Lectures on Modular Forms*, Lecture notes, Princeton University Press (1962).
- [24] Z. Nehari, *Conformal Mapping*, 1st Edition, McGraw-Hill, N.Y. (1952).
- [25] H. Cohn, *Conformal Mapping on Riemann Surfaces*, Dover, N.Y. (1967).
- [26] H. Frakas and I. Kra, *Riemann Surfaces*, Springer-Verlag, N.Y. (1980).
- [27] N. Dorey, V.V. Khoze and M.P. Mattis, hep-th/9606199.
- [28] P.S. Howe and P.C. West, hep-th/9607239.
- [29] A. Bilal, hep-th/9601007.
- [30] W. Nahm, hep-th/9608121; A. Brandhuber and S. Stieberger, hep-th/9609130.